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NODAL GEOMETRY ON RIEMANNIAN MANIFOLDS

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1. Let M^n be a smooth, compact, and connected Riemannian manifold with no boundary. Let Δ denote the Laplacian on M. Let $-\Delta u = \lambda u$, u an eigenfunction with eigenvalue λ, $\lambda > 1$. Our main theorems are:

Theorem 1 (BMO estimate for $\log|u|$). *For u, λ as above,*

$$\| \log |u| \|_{\mathrm{BMO}} \leq c\lambda^n \log \lambda ,$$

where c is independent of λ, and depends only on n and M.

Theorem 2 (Geometry of nodal domains). *Let u, λ be as above, let $B \subset M$ be any ball, and let $\Omega \subset B$ be any of the connected components of $\{x \in B : u(x) \neq 0\}$. If Ω intersects the middle half of B, then*

$$|\Omega| \geq c\lambda^{-2n^2-n/2}(\log \lambda)^{-2n}|B| ,$$

where c is independent of λ and u.

Similar theorems have been proved by H. Donnelly and C. Fefferman [1], [2] with $\lambda^n \log \lambda$ replaced by $\lambda^{n(n+2)/4}$ in Theorem 1 and $\lambda^{-2n^2-n/2}(\log \lambda)^{-2n}$ replaced by $\lambda^{-(n+n^2(n+2))/2}$ in Theorem 2. Of course, it is obvious that Theorems 1 and 2 above are not best possible.

Theorem 1 is the key to Theorem 2. We deduce Theorem 2 from Theorem 1 by essentially following the arguments in [2] with appropriate modifications in view of the better BMO estimate of Theorem 1.

We shall use the symbols c, c_0, c_1, c_2, c_3, c_4, and $\bar{c}$ to denote generic constants which are independent of λ.

2. Before commencing the proof of Theorem 1, we recall two facts from [2]. We state these as Theorem 0.

Theorem 0. *Let M, u, λ be as above. Let $B(x, \delta)$ denote the ball centered at x of radius δ. Then*

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(A) $$\int_{B(x,\delta(1+\lambda^{-1/2}))} |u|^2 \le c \int_{B(x,\delta)} |u|^2,$$

(B) $$\left(\int_{B(x,\delta)} |\nabla u|^2\right)^{1/2} \le c\frac{\sqrt{\lambda}}{\delta} \left(\int_{B(x,\delta)} |u|^2\right)^{1/2}.$$

We now begin the proof of Theorem 1.

Lemma 1. *Let u, λ be as before. Then u satisfies the reverse-Holder inequality,*

$$\left(\frac{1}{|B|}\int_B |u|^{2n/(n-2)}\right)^{(n-2)/2n} \le c\sqrt{\lambda} \left(\frac{1}{|B|}\int_B |u|^2\right)^{1/2}.$$

Proof. By the Poincaré-Sobolev inequality, for any ball B,

$$\left(\frac{1}{|B|}\int_B \left|u - \frac{1}{|B|}\int_B u\right|^{2n/(n-2)}\right)^{(n-2)/2n} \le c|B|^{1/n} \left(\frac{1}{|B|}\int_B |\nabla u|^2\right)^{1/2}.$$

We now apply Theorem 0(B) to the right side above, to get

$$\left(\frac{1}{|B|}\int_B \left|u - \frac{1}{|B|}\int_B u\right|^{2n/(n-2)}\right)^{(n-2)/2n} \le c\sqrt{\lambda} \left(\frac{1}{|B|}\int_B |u|^2\right)^{1/2}.$$

By use of Minkowski's inequality, Lemma 1 follows. q.e.d.

Our theorem will follow from the lemma stated below.

Lemma 2. *Suppose $w > 0$,*

(2.1) $$\int_{B(x,\delta(1+1/\sqrt{\lambda}))} w \le c_0 \int_{B(x,\delta)} w,$$

and

(2.2) $$\left(\frac{1}{|B|}\int_B w^{n/(n-2)}\right)^{(n-2)/n} \le c_1\lambda \frac{1}{|B|}\int_B w.$$

Then $\|\log w\|_{\mathrm{BMO}} \le c(n)\lambda^n \log\lambda$.

Theorem 1 follows by choosing $w = |u|^2$.

Our next lemma is a covering lemma of independent interest.

Lemma 3. *Fix any $\delta > 0$, with $\delta < 1/2$. Let $\{B_\alpha\}_{\alpha\in I}$ be any finite collection of balls in $\mathbb{R}^n$. Then one can find a subcollection of balls $B_1, B_2, \cdots, B_N$ such that*

(a) $$\bigcup_{\alpha\in I} B_\alpha \subset \bigcup_{i=1}^{N} (1+\delta)B_i,$$

(b) $$\sum_{i=1}^{N} \chi_{B_i}(x) \le 4^n\delta^{-n}.$$

Proof. Select a ball B_1 with the largest radius from the collection $\{B_\alpha\}_{\alpha\in I}$. Having selected $B_1, \cdots, B_{k-1}$ select B_k so that $B_k \not\subset \bigcup_{i=1}^{k-1}(1+\delta)B_i$ and B_k has the largest possible radius out of the balls in the collection $\{B_\alpha\}_{\alpha\in I}\backslash\{B_i\}_{i=1}^{k-1}$. Our choice of the subcollection $B_1, \cdots, B_N$ clearly satisfies (a). We now prove (b). Let $x_0 \in \bigcap_{i=1}^{M} B_i$, $M = M(x_0)$. By a translation we may suppose $x_0 = 0$. For $x \in \mathbb{R}^n$, define $T_r(x) = x/r$, $r > 0$. By our selection procedure $T_{r_k}(B_k) \not\subset \bigcup_{i=1}^{k-1}(1+\delta)T_{r_k}(B_i)$, where r_i denotes the radius of B_i. Now $T_{r_k}(B_i)$ is also a ball containing the origin, and since $r_i \geq r_k$ for $i < k$, we have $T_{r_i}(B_i) \subset T_{r_k}(B_i)$, and we conclude

$$(*) \qquad T_{r_k}(B_k) \not\subset \bigcup_{i=1}^{k-1}(1+\delta)T_{r_i}(B_i).$$

Let z_i denote the center of $T_{r_i}(B_i)$. We note that each of the balls $T_{r_i}(B_i)$ has radius 1 and $0 \in \bigcap_{i=1}^{M} T_{r_i}(B_i)$. We will show that $|z_i - z_j| \geq \delta$. For if $|z_i - z_j| < \delta$, and assuming $r_i \geq r_j$, we get $(1+\delta)T_{r_i}(B_i) \supset T_{r_j}(B_j)$, a violation of $(*)$. Thus the balls $B(z_i, \delta/2)$ are all disjoint. Furthermore $T_{r_i}(B_i) \subset \{x : |x| < 2\}$ for all $i = 1, 2, \cdots, M$. Hence, $M(\delta/2)^n \leq 2^n$, i.e., $M \leq 4^n\delta^{-n}$, and (b) follows.

Lemma 4. *Let w satisfy the hypothesis of Lemma 2, let B be a fixed ball, and let $E \subset B$ such that $|E| \geq (1 - c_2\lambda^{-n})^k|B|$. Then*

$$\int_E w \geq (c_3\lambda^{-n/2})^k \int_B w,$$

where $c_2 = c_2(n, c_1)$ and $c_3 = c_3(n, c_0)$.

Proof. The proof of Lemma 4 rests on an induction on k, the inductive step being accomplished by Lemma 3. We verify Lemma 4 for $k = 1$. To do so note that if $|E| \geq (1 - \bar{c}\lambda^{-n/2})|B|$ (for some appropriate choice of $\bar{c} = \bar{c}(c_1, n)$), then $\int_E w \geq \frac{1}{2}\int_B w$. To see this, observe $|B\backslash E| \leq \bar{c}\lambda^{-n/2}|B|$. Thus by (2.2),

$$\int_{B\backslash E} w \leq \left(\int_B w^{n/(n-2)}\right)^{(n-2)/n} |B\backslash E|^{2/n} \leq \bar{c}^{2/n}c_1 \int_B w.$$

We make the choice $\bar{c}^{2/n}c_1 < 1/2$ and inserting this choice into the inequality above we get $\int_{B\backslash E} w \leq \frac{1}{2}\int_B w$. Thus $\int_E w \geq \frac{1}{2}\int_B w$. If $c_2 \leq \bar{c}$ and $|E| \geq (1 - c_2\lambda^{-n})|B|$, then $|E| \geq (1 - \bar{c}\lambda^{-n/2})|B|$. Therefore $\int_E w \geq \frac{1}{2}\int_B w \geq c_3\lambda^{-n/2}\int_B w$, and we are done with the case $k = 1$.

So we assume the statements are valid for $k-1$. Clearly we can assume $|E| \le (1-\bar{c}\lambda^{-n/2})|B|$ or else there is nothing to prove. For each point of density x of E we can thus select a ball $B_x \subset B$ such that $x \in B_x$ and

$$|B_x \cap E|/|B_x| = 1 - \bar{c}\lambda^{-n/2}.$$

We apply the covering lemma, Lemma 3, to the B_x's with the choice $\delta = \lambda^{-1/2}$, and also assume without loss of generality that the B_x are finitely many. Define $E_1 = (\bigcup_{i=1}^N (1+\lambda^{-1/2})B_i) \cap B$. Then $E_1 \subset B$, and to complete the induction we will show that

$$|E| \le (1 - c_2\lambda^{-n})|E_1|, \tag{2.3}$$

$$\int_E w \ge c_3\lambda^{-n/2}\int_{E_1} w. \tag{2.4}$$

We prove (2.3) first. Now,

$$\begin{aligned} |E_1| &= |E| + \left|\left(\bigcup (1+\lambda^{-1/2})B_i\right) \cap B \backslash E\right| \\ &\ge |E| + \left|\left(\bigcup B_i\right) \cap B\backslash E\right| = |E| + \left|\bigcup B_i \backslash E\right|. \end{aligned}$$

By the covering lemma, Lemma 3, the expression above is bounded below by

$$|E| + 4^{-n}\lambda^{-n/2}\sum |B_i\backslash E|. \tag{2.5}$$

By our selection, $|B_i\backslash E| = \bar{c}\lambda^{-n/2}|B_i|$, thus (2.5) is bounded below by

$$|E| + \sum \bar{c}4^{-n}\lambda^{-n}|B_i| = |E| + \bar{c}4^{-n}(1+\lambda^{-1/2})^{-n}\lambda^{-n}\sum |(1+\lambda^{-1/2})B_i|.$$

Set $\bar{c}4^{-n}(1+\lambda^{-1/2})^{-n} = c_2$, and note $c_2 \le \bar{c}$. From the expression above we deduce

$$|E_1| \ge c_2\lambda^{-n}|E_1| + |E|,$$

and (2.3) follows.

We now prove (2.4). By (2.1),

$$\int_{E_1} w \le \sum \int_{(1+\lambda^{-1/2})B_i} w \le c_0 \sum \int_{B_i} w.$$

But $(1-\bar{c}\lambda^{-n/2})|B_i| = |E \cap B_i|$, thus $\int_{B_i} w \le 2\int_{B_i \cap E} w$. Therefore, by Lemma 3,

$$c_0 \sum \int_{B_i} w \le 2c_0 \sum \int_{B_i \cap E} w \le 2c_0 \int_E w \sum \chi_{B_i} \le 2\cdot 4^n c_0 \lambda^{n/2} \int_E w.$$

We select $c_3^{-1} = 2\cdot 4^n c_0$ and (2.4) follows. q.e.d.

We now prove Lemma 2. It will be enough to assume $|B|^{-1}\int_B w = 1$, and to show for $t > 0$

$$|\{x \in B : w^{-1}(x) > t\}| \leq \frac{|B|}{t^{c\lambda^{-n}(\log\lambda)^{-1}}},$$

which is equivalent to showing

$$|\{x \in B : w(x) < t\}| \leq t^{c\lambda^{-n}(\log\lambda)^{-1}}|B|.$$

Let us denote by E the set $\{x \in B : w(x) < t\}$. Select k_0 such that $|E| \sim (1 - c_2\lambda^{-n})^{k_0}|B|$. Thus $k_0 \sim c\lambda^n \log(|B|/|E|)$, and so by Lemma 4 and the normalization $|B|^{-1}\int_B w = 1$, we get,

$$|B| = \int_B w \leq (c_3\lambda^{n/2})^{k_0}\int_E w \leq (c_3\lambda^{n/2})^{k_0} t|E|.$$

Hence,

$$|B|/|E| \leq te^{ck_0 \log\lambda^{n/2}} \leq t(|B|/|E|)^{c\lambda^n \log\lambda},$$

and it follows easily that $|E| \leq t^{c\lambda^{-n}(\log\lambda)^{-1}}|B|$. q.e.d.

We now prove Theorem 2. We will be brief and only indicate those points in our argument which differ substantially from the argument presented in [2]. Before commencing we note an equivalent formulation of Theorem 1:

Theorem 1′. *Let u, λ be as before and let $E \subset B$. Then*

$$\sup_B |u| \leq (c|B|/|E|)^{c\lambda^n \log\lambda} \sup_E |u|.$$

The lemma stated below is proved in [2] (Lemma 2 there).

Lemma 5. *Suppose Ω is a component of $\{x \in B(x_0, \delta), u(x) > 0\}$ and assume $x_0 \in \Omega$ and $0 < \delta < \lambda^{-1/2}$. Suppose further $|\Omega|/|B(x_0, \delta)| < \eta^n < \frac{1}{2}\eta_0^n < \frac{1}{2}$. Then there is a positive number r_0 satisfying*

(a) $$0 < r_0 < \frac{\eta}{\eta_0}\delta,$$

(b) $$\frac{|\Omega \cap B(x_0, r_0)|}{|B(x_0, r_0)|} \geq \eta_0^n,$$

(c) $$\sup_{\Omega \cap B(x_0, r_0)} |u| \leq \left(\frac{r_0}{\delta}\right)^{c_4/\eta} \sup_{B(x_0, \delta)} |u|,$$

where c_4 depends on the "bounded geometry" estimates.

We also need the estimate below which is Theorem 1 in [1].

Lemma 6. *Let u, λ be as above. Then*

$$\sup_{B(x,r)} |u| \leq \left(c\frac{r}{r'}\right)^{c\sqrt{\lambda}} \sup_{B(x,r')} |u|, \qquad 0 < r' < r.$$

Lemma 6 may be deduced from Theorem 0(A) above by an iteration and a use of the mean value inequalities for u.

Proof of Theorem 2. By Theorem $1'$,

$$\sup_{B(x_0,r_0)} |u| \leq \left(\frac{c|B(x_0, r_0)|}{|\Omega \cap B(x_0, r_0)|}\right)^{c\lambda^n \log \lambda} \sup_{B(x_0,r_0)\cap\Omega} |u|$$
$$\leq (c\eta_0^{-n})^{c\lambda^n \log\lambda} \sup_{B(x_0,r_0)\cap\Omega} |u|.$$

The estimate above follows by assuming $|\Omega|/|B(x_0; \delta)| \leq \eta^n$ and then using Lemma 5(b). We shall arrive at a contradiction for a suitable choice of η, and thus for this choice of η we will have $|\Omega|/|B(x_0, \delta)| \geq \eta^n$ which will prove Theorem 2. Using Lemma 5(c) we get

$$(c\eta_0^{-n})^{c\lambda^n \log\lambda} \sup_{B(x_0,r_0)\cap\Omega} |u| \leq (c\eta_0^{-n})^{c\lambda^n \log\lambda} \left(\frac{r_0}{\delta}\right)^{\delta/\eta} \sup_{B(x_0,\delta)} |u|.$$

Thus,

$$\sup_{B(x_0,r_0)} |u| \leq (c\eta_0^{-n})^{c\lambda^n \log\lambda} \left(\frac{r_0}{\delta}\right)^{c_4/\eta} \sup_{B(x_0,\delta)} |u|.$$

Applying Lemma 6 to the left side above yields

$$\sup_{B(x_0,\delta)} |u| \leq \left(c\frac{\delta}{r_0}\right)^{c\sqrt{\lambda}} (c\eta_0^{-n})^{c\lambda^n \log\lambda} \left(\frac{r_0}{\delta}\right)^{c_4/\eta} \sup_{B(x_0,\delta)} |u|$$
$$\leq \left(c\frac{r_0}{\delta}\right)^{c_4/\eta - c\sqrt{\lambda}} (c\eta_0^{-n})^{c\lambda^n \log\lambda} \sup_{B(x_0,\delta)} |u|.$$

Therefore $(cr_0/\delta)^{c_4/\eta - c\sqrt{\lambda}} (c\eta_0^{-n})^{c\lambda^n \log\lambda} \geq 1$.

Let us assume that our choice of η is such that $c_4/\eta - c\sqrt{\lambda} \geq 0$. So using Lemma 5(a) we see easily

$$1 \leq \left(c\frac{\eta}{\eta_0}\right)^{c_4/\eta - c\sqrt{\lambda}} (c\eta_0^{-n})^{c\lambda^n \log\lambda}.$$

We now choose $\eta = \eta_0^2$ and $\eta_0 = \tilde{c}\lambda^{-n}(\log\lambda)^{-1}$. This choice forces $c_4/\eta - c\sqrt{\lambda} \geq 0$ and also yields

$$1 \leq (c\eta_0)^{c_4/\eta - c\sqrt{\lambda} - cn\lambda^n \log\lambda}.$$

This is a contradiction as $c\eta_0 < 1$ for small $\tilde{c}$. Thus, $|\Omega|/|B(x_0, \delta)| > \eta^n = c\lambda^{-2n}(\log\lambda)^{-2n}$. We now get rid of the restriction $\delta < \lambda^{-1/2}$. Suppose $B \subset M$ is any ball with radius $r > \lambda^{-1/2}$. Assume $x_0 \in \Omega$ belongs also to the middle half of B. We apply our previous conclusion to $\Omega \cap B(x_0, \lambda^{-1/2})$ to get

$$|\Omega \cap B| \geq |\Omega \cap B(x_0, \lambda^{-1/2})| \geq c\lambda^{-2n^2}(\log\lambda)^{-2n}|B(x_0, \lambda^{-1/2})| = \frac{c\lambda^{-2n^2-n/2}(\log\lambda)^{-2n}}{r^n}|B|.$$

But $r \leq c_0$ as the manifold is compact and we hence arrive at

$$|\Omega \cap B| \geq c\lambda^{-2n^2-n/2}(\log\lambda)^{-2n}|B|.$$

References

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